Extended black holes in strong gravitational waves

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ABSTRACT

We describe a large class of solutions in pure gravity, dilaton gravity and supergravity corresponding to extended higher-dimensional black holes with strong (non-linear) gravitational waves propagating along their worldvolume. For pure gravity, the extended black holes are higher-dimensional analogs of the point-like Schwarzschild black hole in four dimensions. For supergravity, they are non-extremal p-branes. The gravitational waves can be both space-filling and localized around the worldvolume of the extended black holes. The solutions we present contain a large number of arbitrary functions of the light-cone time describing the amplitudes of different non-linear gravitational wave modes.

1 Introduction

Exact interactions of strongly non-linear objects are generally difficult to describe, and the cases when an analytic treatment can be given are rare. In a gravitational theory, such results give an exact description of strong gravitational effects. Here, we shall present solutions in pure gravity, dilaton gravity and supergravity corresponding to extended black holes with strong gravitational waves propagating along their worldvolume. There are not many known solutions for black holes in time-dependent (or cosmological) backgrounds (examples include black holes in Friedmann-Robertson-Walker [1, 2] and de-Sitter [3, 4] spacetimes, as well as supergravity p-branes embedded in dilaton cosmologies [5]). A specific feature of our solutions is that they contain a large number of arbitrary functions of the light-cone time corresponding to the profiles of the various polarization components of the strong gravitational waves involved.

Strong gravitational waves in flat space-time are known to be described by the metric (see, e.g., Appendix A of [6])

$$ds^2 = -2du\,dv + K_{ij}(u)x^i x^j du^2 + (dx^i)^2, \tag{1}$$

where $K_{ij}(u)$ represent the profiles of different polarization components of the wave. In pure gravity, $K_{ij}(u)$ is constrained by $K_{ii} = 0$, giving the same number of polarizations as in linearized theory (a traceless symmetric tensor in $D-2$ dimensions, with D being the number of dimensions of space-time). If a dilaton is present, K_{ii} does not vanish and is related to the dilaton, which gives an additional independent polarization component. The metric (1) is given in the so-called Brinkmann coordinates (which are not prone to coordinate singularities and hence useful for global considerations). By a u-dependent rescaling of x^i, it can be brought to the so-called Rosen form, in which the metric only depends on u, making the planar nature of the wave front manifest.

In this paper, we shall consider the (considerably more complex) analogs of (1), in which the strong gravitational wave propagates along an extended black hole. The black hole can be an extended higher-dimensional Schwarzschild-like object in pure gravity, or its dilaton gravity modification, or a non-extremal p-brane in supergravity. Because of the presence of the black hole, our solution will display two kinds of non-linear modes: those approaching (1) far away from the black hole, and those localized near the extended black hole worldvolume.

Simpler supersymmetric analogs of our present solutions have been previously considered in a series of publications [7]–[10] with specific choices of the wave profile (for some related literature, see [11]–[19]). Extremal supersymmetric solutions with an arbitrary gravitational wave profile were constructed in [20], and solutions with an arbitrary profile featuring intersecting p-branes were constructed in [21]. Here, we generalize the previous derivations to the case of non-extremal non-supersymmetric extended black holes along the lines suggested in [20].

The paper is organized in a straightforward manner: we first present the general equations of motion and discuss the structure that permits their thorough analysis. We then proceed with the more computationally transparent case of dilaton gravity (that includes pure gravity). Finally, we present the analysis of the non-extremal supergravity p-branes.

2   Equations of motion

We start with the low-energy effective action for the supergravity system coupled to dilaton and n_A-form field strengths (pure gravity and dilaton gravity are obtained straightforwardly by setting some of the fields to zero):

$$I = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - \sum_{A=1}^{m} \frac{1}{2n_A!} e^{a_A\phi} F_{n_A}^2 \right], \tag{2}$$

where G_D is the Newton constant in D dimensions and g is the determinant of the metric. The last term includes both RR and NS-NS field strengths, and $a_A = \frac{1}{2}(5 - n_A)$ for RR field strength and $a_A = -1$ for NS-NS 3-form. We put fermions and other background fields to be zero.

From the action (2), one can derive the field equations

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \sum_A \frac{1}{2n_A!} e^{a_A\phi} \left[n_A \left(F_{n_A}^2 \right)_{\mu\nu} - \frac{n_A - 1}{D - 2} F_{n_A}^2 g_{\mu\nu} \right], \tag{3}$$

$$\Box\phi = \sum_A \frac{a_A}{2n_A!} e^{a_A\phi} F_{n_A}^2, \tag{4}$$

$$\partial_{\mu_1} \left(\sqrt{-g} e^{a_A\phi} F^{\mu_1\cdots\mu_{n_A}} \right) = 0, \tag{5}$$

where $F_{n_A}^2$ denotes $F_{\mu_1\cdots\mu_{n_A}} F^{\mu_1\cdots\mu_{n_A}}$ and $(F_{n_A}^2)_{\mu\nu}$ denotes $F_{\mu\rho\cdots\sigma} F_\nu{}^{\rho\cdots\sigma}$. The Bianchi identity for the form field is given by

$$\partial_{[\mu} F_{\mu_1\cdots\mu_{n_A}]} = 0. \tag{6}$$

In this paper we consider a general D-dimensional theory (which includes the ten-dimensional supergravity case) and assume the following metric form:

$$ds_D^2 = e^{2A(u,r)} \left[-2dudv + K(u,r,y^\alpha)du^2 \right] + e^{2C(u,r)}(dy^\alpha)^2 + e^{2B(u,r)} \left(dr^2 + r^2 d\Omega_{q+1}^2 \right), \tag{7}$$

where the coordinates u, v and y^α, $(\alpha = 1, \ldots, p-1)$ parameterize the $(p+1)$-dimensional worldvolume of the extended black hole, and the remaining $(q + 2)$ coordinates r and angles are transverse to the brane worldvolume, $p + q + 3 = D$, $d\Omega_{q+1}^2$ is the line element of the $(q + 1)$-dimensional sphere. If the y^α-dependence of K is removed in (7), the metric represents the most general space-time with rotational symmetries of x^a and y^α and translational symmetries of v and y^α (with the v-translation being light-like). It turns out that the y^α-dependence can be added to K with only minimal modifications to the equations of motion (and it is useful for including y^α-polarizations of the space-filling plane waves). Such an ansatz has already been considered in [10] for supersymmetric solutions. All the remaining components of the metric and the dilaton are assumed to be functions of u and r only.

For the background RR-form, we take

$$F_{p+2} = E'(u,r)du \wedge dv \wedge dy^1 \wedge \cdots \wedge dy^{p-1} \wedge dr, \qquad n = p + 2, \quad a = \frac{3 - p}{2}. \tag{8}$$

Throughout this paper, the prime and dot denote the derivatives with respect to r and u, respectively, and a, b are angular coordinates. We could also include magnetic background in the same form as the electric one. The result gives basically the same set of equations with minor modifications.

The Einstein and dilaton equations are given as [10]

$$(p-1)(\ddot{C} - 2\dot{A}\dot{C} + \dot{C}^2) + (q+2)(\ddot{B} - 2\dot{A}\dot{B} + \dot{B}^2) + e^{2(A-B)}\Big[KA'' + \frac{1}{2}K''$$

$$+\Big(KA' + \frac{1}{2}K'\Big)\Big(U' + \frac{q+1}{r}\Big)\Big] + \frac{1}{2}e^{2(A-C)}\partial_\alpha^2 K = -\frac{1}{2}\dot{\phi}^2 + \frac{q}{16}KS(E')^2 e^{2(A-B)}, \quad (9)$$

$$\dot{A}' + (p-1)(\dot{C}' - \dot{C}A' - C'\dot{B} + \dot{C}C') + (q+1)\dot{B}' - (q+2)A'\dot{B} = -\frac{1}{2}\dot{\phi}\phi', \quad (10)$$

$$A'' + \Big(U' + \frac{q+1}{r}\Big)A' = \frac{q}{16}S(E')^2, \quad (11)$$

$$C'' + \Big(U' + \frac{q+1}{r}\Big)C' = \frac{q}{16}S(E')^2, \quad (12)$$

$$U'' + B'' - \Big(2A' + (p-1)C' - \frac{q+1}{r}\Big)B' + 2A'^2 + (p-1)C'^2$$

$$= -\frac{1}{2}\phi'^2 + \frac{q}{16}S(E')^2, \quad (13)$$

$$B'' + \Big(U' + \frac{q+1}{r}\Big)B' + \frac{U'}{r} = -\frac{p+1}{16}S(E')^2, \quad (14)$$

$$e^{-U}r^{-(q+1)}(e^U r^{q+1}\phi')' = \frac{p-3}{4}\epsilon S(E')^2, \quad (15)$$

$$(r^{q+1}e^U SE')' = (r^{q+1}e^U SE')^{\cdot} = 0, \quad (16)$$

where we have defined

$$U \equiv 2A + (p-1)C + qB, \qquad S \equiv e^{\epsilon(3-p)\phi/2 - 2(2A+(p-1)C)}, \quad (17)$$

and $\epsilon = +1(-1)$ is for electric (magnetic) background. Unlike in [10], we no longer constrain U to depend only on u, but consider general solutions for our ansatz.

The general structure of the equations of motion has been already spelled out in [20]: equations (11)–(14) are exactly identical to those for a u-independent problem, i.e., to the equations for an extended black hole without any gravitational waves. Such equations have been extensively discussed in the literature (a large class of solutions has been presented in [22]). In our context, we should take the most general solution to Eqs. (11)–(14), i.e., a static extended black hole, and then promote all the integration constants to functions of u. The resulting expressions should be substituted into Eq. (10), which will impose some constraints on the u-dependences of the arbitrary functions of u (non-linear wave amplitudes) contained in the solution of Eqs. (11)–(14). Finally, Eq. (9) will determine K (without introducing any further constraints). This structure essentially reduces a large part of our problem to constructing static extended black hole solutions, a thoroughly explored subject.

Implementing this program, we first learn from Eq. (16) that

$$r^{q+1}e^U SE' = c, \quad (18)$$

is a constant (related to the p-form charge). We find that Eq. (11) then gives

$$(e^U r^{q+1} A')' = \frac{q}{16} cE', \tag{19}$$

which can be immediately integrated to yield

$$A' e^U r^{q+1} = \frac{q}{16} cE + f_A(u), \tag{20}$$

where $f_A(u)$ is an arbitrary function of u. We shall introduce similar functions of u below upon integration of the field equations. By the same token, Eqs. (12) and (15) give

$$C' e^U r^{q+1} = \frac{q}{16} cE + f_C(u), \tag{21}$$

$$\phi' e^U r^{q+1} = \frac{p-3}{4} \epsilon cE + f_\phi(u). \tag{22}$$

Also from Eqs. (11), (12) and (14), we get

$$(r^{2q+1} e^U U')' = 0. \tag{23}$$

Thus

$$(e^U)' = \frac{f_U(u)}{r^{2q+1}}. \tag{24}$$

Eq. (14) can then be rewritten as

$$(B' e^U r^{q+1})' + \frac{f_U(u)}{r^{q+1}} = -\frac{p+1}{16} cE', \tag{25}$$

which yields

$$B' e^U r^{q+1} = -\frac{p+1}{16} cE + \frac{1}{q} \frac{f_U(u)}{r^q} + f_B(u), \tag{26}$$

Using the definition of U, Eqs. (20), (21) and (26), we find

$$2 f_A(u) + (p-1) f_C(u) + q f_B(u) = 0. \tag{27}$$

At this point, Eqs. (20), (21), (22) and (26) have to be substituted into (13), which will result in a constraint on the functions of u introduced thus far. Thereafter, one has to analyze (10), which will give ordinary differential equations for the non-linear wave amplitudes, and (9) to determine K. By use of Eq. (11), (9) can actually be simplified to

$$2(p-1)(\ddot{C} - 2\dot{A}\dot{C} + \dot{C}^2) + 2(q+2)(\ddot{B} - 2\dot{A}\dot{B} + \dot{B}^2) + \dot{\phi}^2 = e^{2(A-B)} \frac{(e^U r^{q+1} K')'}{e^U r^{q+1}} + e^{2(A-C)} \partial_\alpha^2 K. \tag{28}$$

It is convenient to enforce the Brinkmann parametrization (1) for the plane wave at $r \to \infty$. Then A, B and C go to 0 for large r. For the asymptotic large r plane wave, it is not difficult to include the $\alpha\beta$ polarizations by assuming

$$K(u, r, y^\alpha) = k(u, r) + K_{\alpha\beta}(u) y^\alpha y^\beta. \tag{29}$$

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Then, Eq. (28) reduces to

$$2(p-1)(\ddot{C}-2\dot{A}\dot{C}+\dot{C}^2)+2(q+2)(\ddot{B}-2\dot{A}\dot{B}+\dot{B}^2)+\dot{\phi}^2 = e^{2(A-B)}\frac{(e^U r^{q+1}k')'}{e^U r^{q+1}}+e^{2(A-C)}K_{\alpha\alpha}. \tag{30}$$

One then simply takes a solution for A, B, C and ϕ (with all the arbitrary functions of u contained therein), specifies $K_{\alpha\beta}(u)$ and solves the above equation for $k(r)$. The arbitrary functions of u contained in our solution are thus $K_{\alpha\beta}(u)$ and the functions of u contained in the solution of (10)–(16) subject to A, B and C going to 0 at large r.

Even though the equation for k cannot be solved in terms of elementary functions due to the complexity of r-dependences in A, B, C, ϕ (exact expressions will be given in the subsequent sections), k can be always expressed as two integrations over r of a given r- and u-dependent quantity. (We could set $k = 0$, but then this would give a constraint on the solution [21].)

Our final step is to give an explicit analysis of (10)–(16). Since the general supergravity case is quite complicated algebraically, we shall first set c to 0 (i.e., assume no p-form charge) and see how this type of derivations works (there is a rather non-trivial interplay between the structure of the constraint equation (13) and the amplitude evolution equation (10)). We shall then proceed with the case of finite charge non-extremal supergravity p-branes.

3   Pure and dilaton gravity

In view of its algebraic simplicity, it seems reasonable to consider the case of vanishing form charge first. This amounts to simply setting $c = 0$ in the equations of the previous section.

One can use (14) and (24) to eliminate second derivatives from (13):

$$-U'^2-\frac{2(q+1)}{r}U'-(2A'+(p-1)C'+U')B'+2A'^2+(p-1)C'^2+\frac{1}{2}\phi'^2=0, \tag{31}$$

or, expressing B through U, A and C:

$$2A'^2+(p-1)C'^2+\frac{(2A'+(p-1)C')^2}{q}+\frac{1}{2}\phi'^2-\frac{U'^2(q+1)}{q}-\frac{2(q+1)U'}{r}=0. \tag{32}$$

We then multiply this equation by $r^{2(q+1)}e^{2U}$ and use (20), (21) and (24) together with

$$e^U=h_U-\frac{f_U}{2qr^{2q}}. \tag{33}$$

The r-dependences in the last two terms of (32) cancel each other, leaving an r-independent constraint equation:

$$2f_A^2+(p-1)f_C^2+\frac{(2f_A+(p-1)f_C)^2}{q}-2(q+1)f_Uh_U+\frac{1}{2}f_\phi^2=0, \tag{34}$$

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or

$$(q+2)f_A^2 + 2(p-1)f_A f_C + \frac{(p-1)(D-4)}{2}f_C^2 + \frac{q}{4}f_\phi^2 - q(q+1)f_U h_U = 0. \qquad (35)$$

(Note that, from (34), $f_U h_U$ is a sum of squares, and thus positive.)

We now turn to Eq. (10). Expressing B through U, A and C, we obtain

$$- (q+2)(\dot{A}' + A'\dot{U}) - (p-1)(\dot{C}' + C'\dot{U}) + 2(q+2)A'\dot{A} + 2(p-1)(A'\dot{C} + C'\dot{A})$$
$$+ \frac{q}{2}\dot{\phi}\phi' + (p-1)(D-4)C'\dot{C} + (q+1)\dot{U}' = 0. \quad (36)$$

We then multiply this equation by $r^{q+1}e^U$ and use (20, 21) together with their u-derivatives. The result is

$$- (q+2)\dot{f}_A - (p-1)\dot{f}_C + 2(q+2)f_A\dot{A} + 2(p-1)(f_A\dot{C} + f_C\dot{A}) + \frac{q}{2}f_\phi\dot{\phi}$$
$$+ (p-1)(D-4)f_C\dot{C} + (q+1)r^{q+1}e^U\dot{U}' = 0. \qquad (37)$$

We then integrate (20, 21) to obtain

$$A = h_A(u) - \frac{\sqrt{2}f_A}{\sqrt{qh_U f_U}} \operatorname{arctanh}\left(\sqrt{\frac{f_U}{2qh_U}}\frac{1}{r^q}\right), \qquad (38)$$

where $h_A(u)$ is an arbitrary function of u, and an analogous expression for C and ϕ. Differentiating these formulas with respect to u yields

$$\dot{A} = \dot{h}_A - \left(\frac{\sqrt{2}f_A}{\sqrt{qh_U f_U}}\right)^{\cdot} \operatorname{arctanh}\left(\sqrt{\frac{f_U}{2qh_U}}\frac{1}{r^q}\right)$$
$$- \frac{f_A}{q}\sqrt{\frac{h_U}{f_U}}\left(\sqrt{\frac{f_U}{h_U}}\right)^{\cdot}\frac{e^{-U}}{r^q}, \qquad (39)$$

and an analogous expression for $\dot{C}$ and $\dot{\phi}$. Note that $\operatorname{arctanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$, and this looks similar to the solution in [7]. Furthermore, using (33), we obtain

$$\dot{U}' = ((e^U)'/e^U)^{\cdot} = -\frac{f_U^2}{r^{2q+1}e^{2U}}\left(\frac{h_U}{f_U}\right)^{\cdot}. \qquad (40)$$

Substituting the above expressions for time derivatives into (37), we find that there are only three different dependences on r present in that equation. The terms independent of r are

$$M \equiv -(q+2)\dot{f}_A - (p-1)\dot{f}_C + 2(q+2)f_A\dot{h}_A + 2(p-1)(f_A\dot{h}_C + f_C\dot{h}_A)$$
$$+ (p-1)(D-4)f_C\dot{h}_C + \frac{q}{2}f_\phi\dot{h}_\phi. \qquad (41)$$

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The terms proportional to $\mathrm{arctanh}\left(\sqrt{\frac{f_U}{2qh_U}}\frac{1}{r^q}\right)$ are

$$-\sqrt{\frac{h_U f_U}{q}}\left(\frac{(q+2)f_A^2 + 2(p-1)f_A f_C + \frac{(p-1)(D-4)}{2}f_C^2 + \frac{q}{4}f_\phi^2}{h_U f_U}\right)^{\cdot}$$

$$\times \mathrm{arctanh}\left(\sqrt{\frac{f_U}{2qh_U}}\frac{1}{r^q}\right), \qquad (42)$$

which vanishes by (35). The terms proportional to $\frac{e^{-U}}{r^q}$ are

$$-\left[\frac{1}{q}\sqrt{\frac{h_U}{f_U}}\left(\sqrt{\frac{f_U}{h_U}}\right)^{\cdot}\left(2(q+2)f_A^2 + 4(p-1)f_A f_C + (p-1)(D-4)f_C^2 + \frac{q}{2}f_\phi^2\right)\right.$$

$$\left.+(q+1)f_U^2\left(\frac{h_U}{f_U}\right)^{\cdot}\right]\frac{e^{-U}}{r^q}, \qquad (43)$$

which again vanishes by (35).

We have therefore shown that (10) reduces to a single ordinary differential equation with respect to u (since all the terms except for (41) vanish when (35) is satisfied):

$$M = 0. \qquad (44)$$

As per discussion under (28), it is convenient to enforce the Brinkmann parametrization for the plane wave at large r, in which case A, B and C should go to 0 for r going to infinity. This translates into $h_A = h_C = 0$, $h_U = 1$. We are then left with five functions $(f_A, f_C, f_U, h_\phi, f_\phi)$ subject to two constraints (35) and (44), and additionally $K_{\alpha\beta}(u)$, which are arbitrary functions of u, as explained under Eq. (30). Of these, $K_{\alpha\beta}(u)$ and h_ϕ represent space-filling gravitational waves (whose large r asymptotics in the Brinkmann form flat-space dilaton-gravity plane wave) and the remaining functions do not contribute to the large r behavior of the metric (and hence represent non-linear waves localized on the extended black hole worldvolume).

For the vanishing p-form charge case (which is what we are considering in this section), the dilaton can be consistently set to zero. In this case, we end up with pure gravity (empty space) solutions. It is interesting that, for the pure gravity case, the space-filling and worldvolume modes completely decouple. Indeed, the only space-filling modes in that case are $K_{\alpha\beta}(u)$ (which are arbitrary functions, not affected by the values of f_A, f_C, f_U). On the other hand, $K_{\alpha\beta}(u)$ do not appear in equations (35) and (44) constraining the behavior of the non-linear modes localized on the worldvolume. (In particular, there are always solutions for which the only u-dependence is contained in the uu-component of the metric of our ansatz, corresponding to zero amplitudes of the worldvolume modes.) There could be deeper symmetry reasons for this surprising decoupling of non-linear modes.

We conclude this section by a summary of the expressions for our solution in a more compact notation. We introduce $R = (f_U/2q)^{1/2q}$, $\alpha = f_A/\sqrt{2qf_U}$, $\gamma = f_C/\sqrt{2qf_U}$, $\omega = f_\phi/\sqrt{2qf_U}$ and rename h_ϕ to h. Then the solution is given by (7) with

$$e^A = \left(\frac{1-(R(u)/r)^q}{1+(R(u)/r)^q}\right)^{\alpha(u)}, \qquad (45)$$

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$$e^C = \left(\frac{1 - (R(u)/r)^q}{1 + (R(u)/r)^q} \right)^{\gamma(u)}, \tag{46}$$

$$e^\phi = e^{h(u)} \left(\frac{1 - (R(u)/r)^q}{1 + (R(u)/r)^q} \right)^{\omega(u)}, \tag{47}$$

$$e^B \equiv e^{U/q} e^{-2A/q - (p-1)C/q} = \left(1 - \left(\frac{R(u)}{r} \right)^{2q} \right)^{1/q} \left(\frac{1 - (R(u)/r)^q}{1 + (R(u)/r)^q} \right)^{-(2\alpha(u) + (p-1)\gamma(u))/q}$$

$$= \frac{(1 - (R(u)/r)^q)^{-(2\alpha(u) + (p-1)\gamma(u) - 1)/q}}{(1 + (R(u)/r)^q)^{-(2\alpha(u) + (p-1)\gamma(u) + 1)/q}}. \tag{48}$$

The functions of u appearing above satisfy

$$2(q+2)\alpha^2 + 4(p-1)\alpha\gamma + (p-1)(D-4)\gamma^2 + \frac{q}{2}\omega^2 = q + 1, \tag{49}$$

$$-(q+2)R^{-q}(\alpha R^q)^\cdot - (p-1)R^{-q}(\gamma R^q)^\cdot + \frac{q}{2}\omega\dot{h} = 0, \tag{50}$$

which follow from (35), (41) and (44). K is then determined by (29) and (30).

4 Supergravity

Having examined the simple case of pure and dilaton gravity, we now turn to the general case with nonvanishing field strength. This set-up can be naturally implemented in ten-dimensional supergravity.

Substituting Eqs. (20), (21), (22) and (26) into (13), we get

$$(q+2)f_A^2 + 2(p-1)f_A f_C + \frac{(p-1)(D-4)}{2} f_C^2 + \frac{q}{4} f_\phi^2 - q(q+1)f_U h_U$$

$$+ \frac{q}{4} \left\{ (cE)^2 + 2 \left[2f_A + (p-1)f_C + \frac{p-3}{4}\epsilon f_\phi \right] cE - e^U r^{q+1} cE' \right\} = 0. \tag{51}$$

Let us solve Eq. (51), which can be written as

$$cE' = Q \left[(cE)^2 + 2\kappa cE - \sigma^2 \right], \tag{52}$$

where we have introduced

$$Q = e^{-U} r^{-(q+1)},$$

$$\sigma^2 = -\frac{4}{q} \left\{ f_A^2(q+2) + 2(p-1)f_A f_C + \frac{(p-1)(D-4)}{2} f_C^2 + \frac{q}{4} f_\phi^2 - q(q+1)f_U h_U \right\} \tag{53}$$

Eq. (52) is a special Riccati equation for cE, and it obviously has a particular solution:[1]

$$cE_0 = -\kappa + \tau, \tag{54}$$

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[1] For a general Ricatti equation $g(x)y' = f_2(x)y^2 + f_1(x)y + f_0(x)$, the general solution is given by

$$y(x) = y_0(x) + \Phi(x) \left[\text{const} - \int \Phi(x) \frac{f_2(x)}{g(x)} dx \right]^{-1},$$

where

$$\kappa = 2f_A + (p-1)f_C + \epsilon\left(\frac{p-3}{4}\right)f_\phi,$$
$$\tau^2 = \kappa^2 + \sigma^2. \tag{55}$$

So the general solution of the Riccati Eq. (52) is given by

$$cE = cE_0 + e^{2\tau w}\left(h_E - \int e^{2\tau w}Qdr\right)^{-1}$$
$$= -\kappa + \tau + \frac{2\tau e^{2\tau w}}{2\tau h_E + 1 - e^{2\tau w}}, \tag{56}$$

where h_E is an arbitrary function of u, as explained in the preceding footnote. Note that Eq. (13) gave a constraint (44) on the u-dependent functions when we set the field strength to zero, but here it determines the field strength instead of constraining the functions.

With the result (56), we find

$$z \equiv \int cEQdr = (-\kappa + \tau)w - \ln\left|\frac{1}{2\tau}\left(2\tau h_E + 1 - e^{2\tau w}\right)\right|. \tag{57}$$

After substituting (57) into Eqs. (20-22), we get the expressions for A, C and ϕ:

$$A = h_A + f_Aw + \frac{q}{16}\left\{(-\kappa + \tau)w - \ln\left|\frac{1}{2\tau}\left(2\tau h_E + 1 - e^{2\tau w}\right)\right|\right\},$$
$$C = h_C + f_Cw + \frac{q}{16}\left\{(-\kappa + \tau)w - \ln\left|\frac{1}{2\tau}\left(2\tau h_E + 1 - e^{2\tau w}\right)\right|\right\},$$
$$\phi = h_\phi + f_\phi w + \epsilon\frac{p-3}{4}\left\{(-\kappa + \tau)w - \ln\left|\frac{1}{2\tau}\left(2\tau h_E + 1 - e^{2\tau w}\right)\right|\right\}, \tag{58}$$

where

$$w \equiv \int Qdr = -\sqrt{\frac{2}{qf_Uh_U}}\,\text{arctanh}\left(\sqrt{\frac{f_U}{2qh_U}}\frac{1}{r^q}\right). \tag{59}$$

Then Eq. (18) gives

$$c^2e^{2N} = 2\tau h_E + 1, \tag{60}$$

where we have defined

$$N \equiv 2h_A + (p-1)h_C + \frac{p-3}{4}\epsilon h_\phi. \tag{61}$$

---

where y_0 is a particular solution and

$$\Phi(x) = \exp\left\{\int\left[2f_2(x)y_0(x) + f_1(x)\right]\frac{1}{g(x)}dx\right\}.$$

In our case, $\Phi = e^{2\tau w}$. We get the particular solution when the integral constant "const" approaches infinity. In our present context, this integration constant can be an arbitrary function of u.

The above equation gives the relation between N and h_E.

Now let us consider the dot-prime equation (10). After substituting

$$\dot{A} = \dot{h}_A + \dot{f}_A w + f_A \dot{w} + \left(\frac{q}{16}\right)\dot{z}, \tag{62}$$

and similar expressions for $\dot{C}$ and $\dot{\phi}$ into Eq. (10), we get

$$M + (q+1)[r^q e^U \dot{U}' + q(f_U h_U)^{\cdot} w + 2q(f_U h_U)\dot{w}]$$
$$-\frac{q}{2}\{c\dot{E} - [(\kappa w)^{\cdot} + \dot{N} + \dot{z}]cE - \kappa\dot{z} + \sigma^2 \dot{w} + \sigma\dot{\sigma}w\} = 0\,, \tag{63}$$

where M and N are defined in Eqs. (41) and (61), respectively. With the help of Eq. (59), Eq. (63) reduces to

$$M - \frac{q}{2}\{c\dot{E} - ((\kappa w)^{\cdot} + \dot{N} + \dot{z})cE - \kappa\dot{z} + \sigma^2\dot{w} + \sigma\dot{\sigma}w\} = 0\,. \tag{64}$$

We also have

$$
\begin{aligned}
c\dot{E} &= \dot{\tau} - \dot{\kappa} + (cE + \kappa - \tau)\left[\frac{\dot{\tau}}{\tau} + (2\tau w)^{\cdot} - \frac{(2\tau h_E + 1 - e^{2\tau w})^{\cdot}}{(2\tau h_E + 1 - e^{2\tau w})}\right], \\
\dot{z} &= (\dot{\tau} - \dot{\kappa})w + (\tau - \kappa)\dot{w} + \frac{\dot{\tau}}{\tau} - \frac{(2\tau h_E + 1 - e^{2\tau w})^{\cdot}}{(2\tau h_E + 1 - e^{2\tau w})}\,,
\end{aligned} \tag{65}
$$

and

$$\frac{(2\tau h_E + 1 - e^{2\tau w})^{\cdot}}{2\tau h_E + 1 - e^{2\tau w}} = \frac{(2\tau h_E)^{\cdot} - (e^{2\tau w})^{\cdot}}{2\tau h_E + 1 - e^{2\tau w}} = \frac{2(2\tau h_E + 1)\dot{N} - (e^{2\tau w})(2\tau w)^{\cdot}}{2\tau h_E + 1 - e^{2\tau w}}\,. \tag{66}$$

Here we have used the relation (60). From (65) and (66), we find

$$c\dot{E} - [(\kappa w)^{\cdot} + \dot{N} + \dot{z}]cE - \kappa\dot{z} = -\dot{\kappa} + (\tau + \kappa)\dot{N} - (\sigma^2\dot{w} + \sigma\dot{\sigma}w)\,, \tag{67}$$

where we have also used $\tau^2 = \kappa^2 + \sigma^2$ and $\tau\dot{\tau} = \kappa\dot{\kappa} + \sigma\dot{\sigma}$. Substituting this result into Eq. (64), we obtain our final form of the constraint (10):

$$M + \frac{q}{2}\{\dot{\kappa} - (\tau + \kappa)\dot{N}\} = 0\,. \tag{68}$$

To summarize, our solutions are given by the metric (7) with

$$
\begin{aligned}
e^{2A} &= e^{2h_A + (f_A + \frac{q}{16}(\tau - \kappa))w}\left(\frac{2\tau}{2\tau h_E + 1 - e^{2\tau w}}\right)^{q/16}, \\
e^{2C} &= e^{2h_C + (f_C + \frac{q}{16}(\tau - \kappa))w}\left(\frac{2\tau}{2\tau h_E + 1 - e^{2\tau w}}\right)^{q/16}, \\
e^{2B} &= e^{(2/q)(U - 2A - (p-1)C)}, \\
e^{2\phi} &= e^{2h_\phi + (f_\phi + \epsilon\frac{p-3}{4}(\tau - \kappa))w}\left(\frac{2\tau}{2\tau h_E + 1 - e^{2\tau w}}\right)^{q/16},
\end{aligned} \tag{69}
$$

where e^U, κ and τ are given in (33) and (55), and

$$e^w = \left(\frac{g_-}{g_+}\right)^{1/\sqrt{2qf_U h_U}}, \qquad g_\pm \equiv 1 \pm \sqrt{\frac{f_U}{2qh_U}\frac{1}{r^q}}, \tag{70}$$

and use has been made of the relation $\mathrm{arctanh}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. The functions $g_\pm$ are time-dependent harmonic functions. The u-dependent functions $h_A, h_C, h_\phi, f_A, f_C, f_\phi$ and h_E are constrained by (60) and (68).

In conclusion, we consider a few special cases representing dynamically invariant subspaces in the space of non-linear gravitational wave modes.

4.1 $\sigma = 0$ and $\kappa \neq 0$

In this case, we have $\tau - \kappa = 0$, so the solution A, C and ϕ becomes

$$
\begin{aligned}
A &= h_A + f_A w - \frac{q}{16}\ln\left|\frac{1}{2\kappa}\left(2\kappa h_E + 1 - e^{2\kappa w}\right)\right|, \\
C &= h_C + f_C w - \frac{q}{16}\ln\left|\frac{1}{2\kappa}\left(2\kappa h_E + 1 - e^{2\kappa w}\right)\right|, \\
\phi &= h_\phi + f_\phi w - \epsilon\frac{p-3}{4}\ln\left|\frac{1}{2\kappa}\left(2\kappa h_E + 1 - e^{2\kappa w}\right)\right|.
\end{aligned}
\tag{71}
$$

Eq. (68) becomes

$$M + \frac{q}{2}[\dot{\kappa} - 2\kappa\dot{N}] = 0. \tag{72}$$

4.2 $\kappa = 0$ and $\sigma \neq 0$

In this case we have $\tau = \sigma$, and the functions A, C and ϕ are given by

$$
\begin{aligned}
A &= h_A + f_A w + \frac{q}{16}\left\{\sigma w - \ln\left|\frac{1}{2\sigma}\left(2\sigma h_E + 1 - e^{2\sigma w}\right)\right|\right\}, \\
C &= h_C + f_C w + \frac{q}{16}\left\{\sigma w + \ln\left|\frac{1}{2\sigma}\left(2\sigma h_E + 1 - e^{2\sigma w}\right)\right|\right\}, \\
\phi &= h_\phi + f_\phi w + \epsilon\frac{p-3}{4}\left\{\sigma w + \ln\left|\frac{1}{2\sigma}\left(2\sigma h_E + 1 - e^{2\sigma w}\right)\right|\right\}.
\end{aligned}
\tag{73}
$$

Eq. (68) becomes

$$M - \frac{q}{2}\sigma\dot{N} = 0. \tag{74}$$

4.3 $\tau = 0$

In the $\tau \to 0$ limit, the solution (58) reduces to

$$
\begin{aligned}
A &= h_A + f_A w - \frac{q}{16}\left\{\kappa w + \ln|h_E - w|\right\}, \\
C &= h_C + f_C w - \frac{q}{16}\left\{\kappa w + \ln|h_E - w|\right\}, \\
\phi &= h_\phi + f_\phi w - \epsilon\frac{p-3}{4}\left\{\kappa w + \ln|h_E - w|\right\}.
\end{aligned}
\tag{75}
$$

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Eq. (60) does not give any constraint on h_E. The dot-prime equation reduces to

$$M + \frac{q}{2}\dot{\kappa} = 0 \,. \tag{76}$$

4.4 $\kappa = 0$ and $\sigma = 0$

This special case is very simple. The function A, C and ϕ are given by

$$A = h_A + f_A w - \frac{q}{16}\ln|h_E - w| \,,$$
$$C = h_C + f_C w - \frac{q}{16}\ln|h_E - w| \,,$$
$$\phi = h_\phi + f_\phi w - \epsilon\frac{p-3}{4}\ln|h_E - w| \,. \tag{77}$$

The dot-prime equation is also simple, which is just $M = 0$. Thus the functions f_A, f_C, f_ϕ, h_A, h_C, h_ϕ, h_E, f_U and h_U are constrained by

$$\sigma = \kappa = N = M = 0 \,. \tag{78}$$

If we also require the solution approaches a Brinkmann form plane wave when $r \to \infty$, we have to require the functions in A and C satisfy additional two relations. So, there are totally six constraints on these functions, and most of them are not independent. In this setting, it is possible to prove that $f_U h_U > 0$ (which is something we have implicitly assumed).

Let us consider a simple example (specifying $D = 10$): We set $h_E = 1$, and $e^A \to 1$ and $e^C \to 1$ gives $h_A = h_C = 0$. The condition $M = 0$ reduces to

$$0 = -(q+2)\dot{f}_A - (p-1)\dot{f}_C + \frac{q}{2}f_\phi h_\phi \,. \tag{79}$$

In addition, f_ϕ and h_ϕ have to satisfy $\kappa = N = 0$, i.e.,

$$0 = 2f_A + (p-1)f_C + \epsilon\frac{p-3}{4}f_\phi \,, \tag{80}$$

$$0 = \epsilon\frac{p-3}{4}h_\phi \,. \tag{81}$$

Further, $\sigma = 0$ gives

$$0 = f_A^2(q+2) + 2(p-1)f_A f_C + 3(p-1)f_C^2 + \frac{q}{4}f_\phi^2 - q(q+1)f_U h_U \,. \tag{82}$$

We now consider all possible cases separately.

(i) For $p = 3$ ($q = 4$), we get $f_A = -f_C$ from (80), and Eq. (82) gives

$$8f_A^2 + f_\phi^2 = 20 f_U h_U > 0 \,. \tag{83}$$

(ii) For $p = 1$ ($q = 6$), we have $h_\phi = 0$ from (81) and then f_A is a constant from (79). Then $f_\phi^2 = 16 f_A^2$ from (80) and Eq. (82) gives

$$16f_A^2 = 21 f_U h_U > 0 \,. \tag{84}$$

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(iii) If $p \neq 3$ and $p \neq 1$, we get $h_\phi = 0$, and

$$s = (q + 2)f_A + (p - 1)f_C \,, \tag{85}$$

is a constant. Substituting this into Eq. (82), we get the second order equation

$$128f_A^2 - 16(p + 1)f_A s + \frac{16p - (p + 1)^2}{q}s^2 = (p - 3)^2(p - 1)(q + 1)f_U h_U \,. \tag{86}$$

Since its discriminant is always negative:

$$\left(16(p + 1)f_A\right)^2 - 4 \times 128f_A^2 \times \left(\frac{16p - (p + 1)^2}{q}\right) = -f_A^2(p - 3)^2(p - 1) < 0 \,, \tag{87}$$

and $16p - (p + 1)^2 > 0$ for $9 \geq p \geq 1$, we always have the left-hand side of Eq. (86) positive for arbitrary s. This means that $f_U h_U > 0$.

Thus in all cases, we have $f_U h_U > 0$ as we have mentioned before.

5 Conclusions

In this paper, starting with an action for form fields and dilaton coupled to gravity, we have derived general solutions describing extended higher-dimensional black holes with strong dilaton-gravity plane waves propagating along their worldvolume. Both space-filling and (localized) worldvolume non-linear wave modes are present in our solutions, giving a large number of arbitrary functions of the light-cone time (corresponding to the non-linear wave amplitudes).

When the field strength is set to zero, the dilaton can also be set to zero and the solution becomes that in pure gravity. It turns out that the space-filling and worldvolume non-linear gravitational wave modes are completely decoupled from each other in that case. We have also obtained solutions with a form field and nontrivial dilaton in a class of theories including ten-dimensional supergravities, and discussed their special cases.

We shall conclude with a few comments on the novelty and possible applications of our solutions. It is quite remarkable that interactions of an extended black hole with a strong gravitational wave can be described exactly, and one might be wondering for the underlying reason that makes the equations solvable. When supersymmetric solutions of [20, 21] were presented, one could be suspecting that supersymmetry linearizes (trivializes) the gravitational dynamics in some sense. We have presently constructed solutions (involving gravitational waves of an arbitrary profile) that break supersymmetry completely. Hence, integrability is not related to supersymmetry in this case, and must have roots in other features of our set-up, most likely, the light-like isometry. The structure of the equations we find appears quite general, and it is shared by a number of different cases (extremal and non-extremal black holes, single and intersecting p-branes). Indeed, we plan to report [23] on non-extremal intersecting brane solutions analogous to the single black p-brane solutions we have described here. For the case of pure gravity, we have presented simple yet non-trivial solutions that appear to have been overlooked in the long development of

that subject (perhaps, because they only exist in more than four space-time dimensions). We have also presented, for the first time to the best of our knowledge, exact solutions involving non-linear oscillations localized in the near-horizon region of extended black holes (solutions of [20, 21] only involved non-linear waves propagating through the entire space-time).

We firmly believe that exact non-linear solutions are of interest in their own right (one can think of the broad and fruitful investigations of black rings and related solutions in higher dimensions, see, e.g., [24]). However, it is worth mentioning that similar set-ups (an extended black object with a wave propagating along its worldvolume) have appeared in a number of string-theoretical considerations. Thus, light-like 'cosmologies' of [25] start with a brane embedded into a singular gravitational wave (the actual papers consider a restricted class of wave profiles, since our general solutions were not available at the time). This system is then used to formulate a time-depended analog of the AdS/CFT correspondence, which relates an AdS-like space-time with a light-like singularity to a gauge theory with light-like time dependences inserted in its Lagrangian. The kind of solutions we have presently developed suggests various generalizations of this construction. Furthermore, oscillations localized on the worldvolume of black objects have appeared in some string-theoretic approaches to Hawking radiation [26]. In the D-brane language, these oscillations are represented by open strings moving along the branes. In the smooth space-time language, the oscillations are nothing but the near-horizon gravitational waves, for which we have presently given a full non-linear description. Constructing intersecting brane generalizations of our solutions will take us even closer to the situation considered in [26].

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